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On the Order of Approximation by Euler and Taylor Means*

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Euler and Taylor means of the Fourier series for functions in the class Lip α, $0 < \alpha < 1$, have been studied by several authors. In this note, the orders of approximation to functions f in this class by either the Euler $(E, 1)$-means or the Taylor means are shown to be of the Jackson order provided that, in each case, a suitable integrability condition is imposed upon the function

$$\varphi_x(t) = \tfrac{1}{2}\{f(x+t) - 2f(x) + f(x-t)\}.$$

PART I

Introduction

Several fundamental properties of (E, q) summability have been discussed in Hardy [2], Knopp [5], Prachar and Schmetterer [7], and Bollinger [1]. Lorch [6] has discussed the Lebesgue constants for $(E, 1)$ summability in 1950. Sufficient conditions for Euler summability were studied by Holland *et al.* [4] in 1975. The degree (order) of approximation by (E, q) means has been discussed by Holland and Sahney [3] in 1976.

For the case $q = 1$, a very precise upper bound will be determined for the degree of approximation by Euler means of the Fourier series for functions $f \in \text{Lip}\ \alpha$, $0 < \alpha < 1$. The L_1 norm of the kernel $K_{n,1}$ of this summability method has been studied by Lorch [6], and since its order is $(2/\pi^2)(\log n + O(1))$ there is no hope to obtain the Jackson order for the error using $(E, 1)$-means, without imposing further conditions.

1. Let $f \in L(-\pi, \pi)$ and be 2π-periodic. Let the Fourier series associated

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with f be given by $S(x)=\sum_{-\infty}^{\infty} c_m e^{imx}$, and its nth partial sum be $S_n(x)=\sum_{m=-n}^{n} c_m e^{imx}$. For each x, write

$$\varphi_x(t) := \tfrac{1}{2}\{f(x+t)-2f(x)+f(x-t)\}. \tag{1.1}$$

Also, for each $q>0$, let $T_{n,q}=T_{n,q}(f;\cdot)$ be the Euler (E,q)-means of S. That is,

$$T_{n,q}(x) := \frac{1}{(1+q)^n}\sum_{m=0}^{n}\binom{n}{m} q^{n-m} S_m(x). \tag{1.2}$$

LEMMA 1.3.

$$T_{n,q}(f;x)=T_{n,q}(x)=\frac{1}{2\pi}\int_{-\pi}^{\pi} K_{n,q}(x-t)f(t)\,dt$$

$$=\frac{1}{2\pi}\int_{-\pi}^{\pi} K_{n,q}(t)f(x-t)\,dt,$$

where

$$K_{n,q}(t)=\left(\frac{q^2+2q\cos t+1}{q^2+2q+1}\right)^{n/2}\frac{\sin\left(n\theta_t+\dfrac{t}{2}\right)}{\sin\dfrac{t}{2}} \tag{1.4}$$

and $\theta_t\in(-\pi,\pi)$ *is uniquely determined by the following relationships*:

$$\sin\theta_t=q\sin(t-\theta_t),\qquad \operatorname{sgn}\theta_t=\operatorname{sgn}t,\qquad |\theta_t|<|t|\leqslant\pi.$$

In particular,

$$K_{n,1}(t)=\cos^n\left(\frac{t}{2}\right)\frac{\sin\left(\dfrac{n+1}{2}\right)t}{\sin\dfrac{t}{2}}\qquad\text{(see Hardy [2]).} \tag{1.5}$$

Furthermore, the error function in approximating f by $T_{n,q}(f;\cdot)$ is given by

$$T_{n,q}(x)-f(x)=\frac{1}{\pi}\int_{0}^{\pi}\varphi_x(t)\,K_{n,q}(t)\,dt. \tag{1.6}$$

Proof. It is well known that S_m can be obtained by taking the convolution of f with the Dirichlet kernel:

$$S_m(x)=\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)\left(\sum_{k=-m}^{m} e^{ikt}\right)dt. \tag{1.7}$$

Hence we have

$$T_{n,q}(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-t) \left\{ \frac{1}{(1+q)^n} \sum_{m=0}^{n} \binom{n}{m} q^{n-m} \sum_{k=-m}^{m} e^{ikt} \right\} dt. \quad (1.8)$$

However, we have, for $t \neq 0$ and $|t| \leqslant \pi$,

$$\begin{aligned} \{\cdots\} &= \frac{1}{(1+q)^n} \sum_{m=0}^{n} \binom{n}{m} q^{n-m} \left[1 + e^{it} \frac{1-e^{imt}}{1-e^{it}} + e^{-it} \frac{1-e^{-imt}}{1-e^{-it}} \right] \\ &= \frac{1}{(1+q)^n} \sum_{m=0}^{n} \binom{n}{m} q^{n-m} \left[\frac{e^{it/2}}{2i \sin \frac{t}{2}} e^{imt} - \frac{e^{-it/2}}{2i \sin \frac{t}{2}} e^{-imt} \right] \\ &= \frac{q^n}{(1+q)^n \sin \frac{t}{2}} Im \left[e^{it/2} (1 + q^{-1} e^{it})^n \right]. \end{aligned}$$

By simple geometry (cf. Fig. 1), this expression can be written as

$$\begin{aligned} \{\cdots\} &= \frac{q^n}{(1+q)^n \sin \frac{t}{2}} (1 + 2q^{-1} \cos t + q^{-2})^{n/2} \sin \left(n\theta_t + \frac{t}{2} \right) \\ &= \left(\frac{q^2 + 2q \cos t + 1}{q^2 + 2q + 1} \right)^{n/2} \cdot \frac{\sin \left(n\theta_t + \frac{t}{2} \right)}{\sin \frac{t}{2}} := K_{n,q}(t) \end{aligned}$$

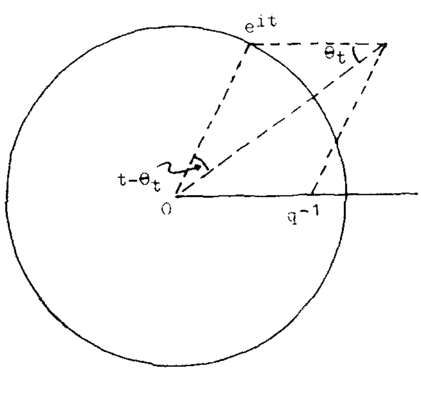

FIGURE 1

with $\sin\theta_t = q\sin(t-\theta_t)$, $\operatorname{sgn}\theta_t = \operatorname{sgn} t$, and $|\theta_t| < t \leqslant \pi$. In particular, if $q=1$, then $\theta_t = t/2$, so that

$$K_{n,1} = \left(\frac{1+\cos t}{2}\right)^{n/2} \frac{\sin\left(\frac{n+1}{2}\right)t}{\sin\frac{t}{2}} = \cos^n\left(\frac{t}{2}\right)\cdot\frac{\sin\left(\frac{n+1}{2}\right)t}{\sin\frac{t}{2}}.$$

Next, we also have

$$\begin{aligned}
&\frac{1}{\pi}\int_0^\pi \varphi_x(t)\,K_{n,q}(t)\,dt\\
&\quad=\frac{1}{2\pi}\int_0^\pi [f(x+t)-2f(x)+f(x-t)]\,K_{n,q}(t)\,dt\\
&\quad=\frac{1}{4\pi}\int_{-\pi}^\pi [f(x+t)+f(x-t)]\,K_{n,q}(t)\,dt\\
&\qquad-f(x)\frac{1}{2\pi}\int_{-\pi}^\pi K_{n,q}(t)\,dt\\
&\quad=T_{n,q}(x)-f(x). \quad \blacksquare
\end{aligned}$$

2. We now study the order of approximation of functions $f \in \operatorname{Lip}\alpha$, $0<\alpha<1$, by the Euler $(E,1)$ means of the Fourier series. We demonstrate in the following theorem that whereas the order of approximation to functions in Lip α, by their Fourier series, is $0(\log n/n^\alpha)$, the order of approximation by $(E,1)$ means of their Fourier series can be reduced to $0(1/n^\alpha)$ provided that a certain integrability condition is imposed upon $\varphi_x(t)$. This gives the optimal order of approximation using Euler $(E,1)$-means.

We have the following theorem:

THEOREM 2.1. *If $f\in\operatorname{Lip}\alpha$, $0<\alpha<1$, is 2π-periodic, and*

$$\int_{2\pi/n}^{\pi} \frac{\left|\varphi_x(t)-\varphi_x\left(t+\frac{2n}{\pi}\right)\right|}{t}\,dt \leqslant Mn^{-\alpha} \tag{2.2}$$

for all x, then

$$E_{n,1}(f) = \|T_{n,1}(f;x)-f(x)\| = 0\left(\frac{1}{n^\alpha}\right), \tag{2.3}$$

where

$$T_{n,1}(f; \cdot) = T_{n,1}(f)$$

is the Euler $(E, 1)$*-means of the Fourier series for* f.

Proof. Using the $(E, 1)$-means of the Fourier series for f, we have

$$\begin{aligned}(T_{n,1} - f)(x) &= \frac{1}{\pi}\int_0^\pi \varphi_x(t)\, K_{n,1}(t)\, dt \\ &= \frac{1}{\pi}\left\{\int_0^{a_n} + \int_{a_n}^{b_n} + \int_{b_n}^{\pi}\right\} \varphi_x(t)\, K_{n,1}(t)\, dt \\ &= \rho_1 + \rho_2 + \rho_3, \qquad \text{say,}\end{aligned}$$

where we write $a_n = (2\pi/n)$ and $b_n = (2\pi/n)^\beta$, $\alpha/(\alpha+1) \leqslant \beta < 1/2$. Now,

$$\begin{aligned}|\rho_1| &\leqslant \frac{1}{\pi}\int_0^{a_n} \frac{|\varphi_x(t)|}{t}\,\frac{nt}{2}\,\frac{\pi}{2}\, dt \\ &= \frac{n}{4}\int_0^{a_n} |\varphi_x(t)|\, dt \\ &\leqslant \frac{n}{4}\cdot M \int_0^{a_n} t^\alpha\, dt \\ &= \frac{M}{4(1+\alpha)}(2\pi)^{1+\alpha}\, n^{-\alpha},\end{aligned}$$

where $|f(x) - f(x+t)| \leqslant Mn^{-\alpha}$, $0 \leqslant M < \infty$. Thus,

$$|\rho_1| = 0(n^{-\alpha}).$$

Also,

$$\begin{aligned}|\rho_3| &\leqslant \frac{2}{\pi}\int_{b_n}^{\pi} \frac{|\varphi_x(t)|}{t}\left|\cos^n\left(\frac{t}{2}\right)\sin\frac{(n+1)\,t}{2}\right| dt \\ &= 0(n^\beta)\cos^n\left\{\frac{1}{2}\left(\frac{2\pi}{n}\right)^\beta\right\}\int_{b_n}^{\pi} |\varphi_x(t)|\, dt \\ &= 0(n^\beta)\left(1 - \frac{1}{4}\,\frac{2^{2\beta}\cdot\pi^{2\beta}}{n^{2\beta}}\right)^n \\ &= 0(n^\beta)\exp\left\{-\frac{2^{2\beta-2}\pi^{2\beta} n}{n^{2\beta}}\right\}\end{aligned}$$

and since $2\beta - 1 < 0$,

$$|\rho_3| = 0(r^{-n}), \qquad r > 1.$$

The study of ρ_2 is more complicated and requires the following calculations. We have

$$\pi\rho_2 = 2\int_{a_n}^{b_n} \frac{\varphi_x(t)}{\sin\frac{t}{2}} \cos^n \frac{t}{2} \sin\frac{(n+1)\,t}{2}\,dt$$

$$= \int_{a_n}^{b_n} \frac{\varphi_x(t)}{\sin\frac{t}{2}} \cos^n \frac{t}{2} \sin\frac{(n+1)\,t}{2}\,dt$$

$$- \int_0^{b_n - a_n} \frac{\varphi_x(t+a_n)}{\sin\frac{(t+a_n)}{2}} \cos^n \left(\frac{t+a_n}{2}\right) \sin\frac{(n+1)\,t}{2}\,dt$$

$$= \int_{a_n}^{b_n} \frac{\varphi_x(t) - \varphi_x(t+a_n)}{\sin\frac{t}{2}} \cos^n \frac{t}{2} \sin\frac{(n+1)\,t}{2}\,dt$$

$$+ \int_{a_n}^{b_n} \frac{\varphi_x(t+a_n)}{\sin\frac{t}{2}} \left[\cos^n \frac{t}{2} - \cos^n\left(\frac{t+a_n}{2}\right)\right] \sin\frac{(n+1)\,t}{2}\,dt$$

$$+ \int_{a_n}^{b_n} \varphi_x(t+a_n) \cos^n \left(\frac{t+a_n}{2}\right)$$

$$\times \left[\frac{1}{\sin\frac{t}{2}} - \frac{1}{\sin\frac{(t+a_n)}{2}}\right] \sin\frac{(n+1)\,t}{2}\,dt$$

$$- \int_0^{a_n} \frac{\varphi_x(t+a_n)}{\sin\frac{(t+a_n)}{2}} \cos^n \left(\frac{t+a_n}{2}\right) \sin\frac{(n+1)\,t}{2}\,dt$$

$$+ \int_{b_n - a_n}^{b_n} \frac{\varphi_x(t+a_n)}{\sin\frac{(t+a_n)}{2}} \cos^n \left(\frac{t+a_n}{2}\right) \sin\frac{(n+1)\,t}{2}\,dt$$

$$= I_1 + I_2 + I_3 + I_4 + I_5.$$

Now,

$$|I_1| \leqslant \int_{a_n}^{b_n} \frac{|\varphi_x(t) - \varphi_x(t+a_n)|}{\sin \frac{t}{2}}\, dt$$

$$\leqslant \pi \int_{a_n}^{b_n} \frac{|\varphi_x(t) - \varphi_x(t+a_n)|}{t}\, dt$$

$$\leqslant Mn^{-\alpha}$$

and

$$I_2 = \int_{an}^{b_n} \frac{\varphi_x(t+a_n)}{\sin \frac{t}{2}} \left[\cos^n \frac{t}{2} - \cos^n \left(\frac{t+a_n}{2}\right)\right] \sin \frac{(n+1)\,t}{2}\, dt$$

$$= \int_{a_n}^{b_n} \frac{\varphi_x(t+a_n)}{\sin \frac{t}{2}} \frac{\pi}{2} \cos^{n-1} \frac{\xi_t}{2} \sin \frac{\xi_t}{2} \sin \frac{(n+1)\,t}{2}\, dt,$$

where $t < \xi_t < t + (2\pi/n) < 2t$. Thus

$$|I_2| \leqslant \pi \int_{a_n}^{b_n} \frac{|\varphi_x(t+a_n)|}{t} \frac{\pi}{2}\, t\, dt$$

$$\leqslant \frac{\pi^2}{2} M \int_{a_n}^{b_n} (t+a_n)^\alpha\, dt$$

$$= 0(b_n^{1+\alpha}) = 0(n^{-\beta(1+\alpha)})$$

$$= 0(n^{-\alpha}).$$

Also,

$$|I_3| \leqslant \int_{a_n}^{b_n} |\varphi_x(t+a_n)| \left| \frac{1}{\sin \frac{t}{2}} - \frac{1}{\sin \frac{t+a_n}{2}} \right| dt$$

$$\leqslant \int_{a_n}^{b_n} \frac{a_n}{2} \frac{|\varphi_x(t+a_n)|}{\sin \frac{t}{2} \sin \frac{t+a_n}{2}}\, dt$$

$$\leqslant M_1 a_n \int_{a_n}^{b_n} \frac{(t+a_n)^\alpha}{t(t+a_n)}\, dt,$$

so that

$$\begin{aligned} |I_3|\, a_n^{-\alpha} &\leqslant M a_n^{1-\alpha} \int_{a_n}^{b_n} \frac{(t+a_n)^\alpha}{t(t+a_n)}\, dt \\ &= M \int_{a_n}^{b_n} \frac{dt}{t\left(\dfrac{t}{a_n}+1\right)^{1-\alpha}} \\ &\leqslant M \int_{a_n}^{\infty} \frac{dt}{t\left(\dfrac{t}{a_n}+1\right)^{1-\alpha}} \\ &= M \int_{1}^{\infty} \frac{1}{t(t+1)^{1-\alpha}}\, dt < \infty. \end{aligned}$$

Consequently,

$$|I_3| \leqslant M_2 n^{-\alpha}.$$

Also,

$$\begin{aligned} |I_4| &\leqslant \pi \int_{a_n}^{2a_n} \frac{|\varphi_x(t)|}{t} \frac{nt}{2}\, dt \\ &= 0(n(2a_n)^{1+\alpha}) \\ &= 0(n^{-\alpha}). \end{aligned}$$

Finally,

$$\begin{aligned} |I_5| &= \left| \int_{b_n-a_n}^{b_n} \left(\frac{\varphi_x(t+a_n)}{\sin \dfrac{t+a_n}{2}} \cos^n \left(\frac{t+a_n}{2}\right) - \sin \frac{nt}{2} \right) dt \right| \\ &\leqslant M \int_0^{a_n} (t+b_n)^{\alpha-1}\, dt \\ &\leqslant M\{(a_n+b_n)^\alpha - b_n^\alpha\}. \end{aligned}$$

However,

$$\begin{aligned} (a_n+b_n)^\alpha - b_n^\alpha &= b_n^\alpha \left\{ \left(1+\frac{a_n}{b_n}\right)^\alpha - 1 \right\} \\ &\leqslant b_n^\alpha \left\{ 1 + \alpha \frac{a_n}{b_n} - 1 \right\} \\ &= \alpha \left(\frac{a_n}{b_n}\right)^{1-\alpha} a_n^\alpha \\ &= 0(a_n^\alpha). \end{aligned}$$

Thus,

$$|I_5| = 0(n^{-\alpha}).$$

Combining $I_1, \ldots, I_5$ we have

$$\rho_2 = 0(n^{-\alpha}), \qquad 0 < \alpha < 1.$$

Consequently, combining ρ_1, ρ_2 and ρ_3, we obtain

$$E_{n,1}(f) = 0(n^{-\alpha}),$$

which was to be proved. ∎

3. Remark: (1) Large "0" in (2.3) can be replaced by small "o", if the corresponding change is made in (2.2).

(2) For $q \neq 1$ and >0, the simple estimates in our proof give the weaker result due to Singh [8] in Lemma 1.3, viz.,

$$(T_{n,q} - f)(x) = 0(n^{-\alpha/2}).$$

PART II

1. Let $\{a_{nk}\}$ be a matrix defined by

$$\frac{(1-r)^{n+1}\,\theta^n}{(1-r\theta)^{n+1}} = \sum_{k=0}^{\infty} a_{nk}\,\theta^k \qquad \text{for} \quad |r\theta| < 1,$$

and n taking only non-negative integer values.

We study, in this section, 2π-periodic functions $f \in L\ [0, 2\pi]$ with a Fourier representation

$$f(x) \approx \sum_{n=0}^{\infty} (a_n \cos nx + b_n \sin nx) \equiv \sum_{n=0}^{\infty} A_n(x),$$

and we let $\varphi_x(t) = \frac{1}{2}\{f(x+t) - 2f(x) + f(x-t)\}$.

DEFINITION 1.1. *A given sequence $\{s_k\}_0^{\infty}$ is said to be Taylor summable, if*

$$\sigma_n^r = \sum_{k=0}^{\infty} a_{nk} s_k$$

tends to a limit as $n \to \infty$, where $0 \leqslant r < 1$.

Several authors, namely, Ishiguro [12], Lorch and Newman [13], and Forbes [10], have studied this method of summability. In 1979, Holland *et al.* [11] found a criterion for Taylor summability of Fourier series.

Let us write $\psi = e^{2iu}$ and $1 - re^{i2u} = \rho e^{-2i\theta}$. Then

$$\begin{aligned}
&\sum_{k=0}^{\infty} a_{nk} \sin(2k+1)\,u \\
&\quad = \operatorname{Im} \sum_{k=0}^{\infty} \{a_{nk} e^{i2ku} e^{iu}\} \\
&\quad = \operatorname{Im} \left\{ \frac{(1-r)^{n+1}\,\psi^n}{(1-r\psi)^{n+1}}\, e^{iu} \right\} \\
&\quad = \operatorname{Im} \left\{ \left(\frac{1-r}{\rho}\right)^{n+1} e^{i(2n+1)u} e^{i2(n+1)\theta} \right\} \\
&\quad = \left(\frac{1-r}{\rho}\right)^{n+1} \sin \left\{ (n+1) \left[2(u+\theta) - \frac{u}{n+1} \right] \right\}.
\end{aligned}$$

However, writing S_k to be the kth partial sum of the Fourier series for f, we have

$$S_k - S = \frac{1}{\pi} \int_0^{\pi} \frac{\varphi_x(t)}{\sin \frac{t}{2}} \sin\left(k + \frac{1}{2}\right) t\, dt;$$

thus,

$$\begin{aligned}
\sigma_n^r &= \sigma_n^r(S_k - S) \\
&= \frac{1}{\pi} \left(\frac{1-r}{\rho}\right)^{n+1} \int_0^{\pi} \frac{\varphi_x(t)}{\sin \frac{t}{2}} \sin \left[(n+1)(t+\theta) - \frac{t}{2}(n+1) \right] dt. \qquad (1.2)
\end{aligned}$$

where now $1 - re^{it} = \rho e^{-i\theta}$.

2. We now have the following theorem:

THEOREM 2.1. *If* $f \in \operatorname{Lip} \alpha$, $0 < \alpha < 1$, *is* 2π*-periodic, and*

$$\int_{a_n}^{\pi} \frac{|\varphi_x(t) - \varphi_x(t + a_n)|}{t}\, dt = 0(n^{-\alpha})$$

uniformly in x, *where* $a_n = \pi\{n + \frac{1}{2} + (n+1)r/(1-r)\}^{-1}$, *then*

$$\|\sigma_n^r - f(x)\| = 0(n^{-\alpha}),$$

where $\sigma_n^r = \sigma_n^r(f; \cdot)$ *is the Taylor mean of order* n *of the Fourier series for* f.

We require the following lemmas.

LEMMA 2.2 [10].

(i) $\left(\frac{1-r}{\rho}\right)^n \leqslant e^{-Ant^2}, \qquad A > 0, 0 \leqslant t \leqslant \pi,$

and

(ii) $\left|\left(\frac{1-r}{\rho}\right)^n - \exp\left(-\frac{nr^2}{2(1-r)^2}\right)\right| \leqslant Bnt^4, \qquad B \text{ constant}, t > 0.$

LEMMA 2.3 [14].

$$\left|\theta - \frac{rt}{1-r}\right| \leqslant ct^3, \qquad 0 \leqslant t \leqslant \frac{\pi}{2}, c \text{ constant}.$$

Proof of the Theorem. Using the Taylor transform of $\{S_k - S\}$, we have

$$\sigma_n^r - f(x)$$

$$= \frac{1}{\pi}\left[\int_0^{a_n} + \int_{a_n}^{b_n} + \int_{b_n}^{\pi}\right] \frac{\varphi_x(t)}{\sin\frac{t}{2}} \left(\frac{1-r}{\rho}\right)^{n+1} \sin\left\{\left(n+\frac{1}{2}\right)t + (n+1)\theta\right\} dt$$

$$= \eta_1 + \eta_2 + \eta_3, \qquad \text{say, where } b_n = a_n^\beta \text{ for } \frac{1+\alpha}{3+\alpha} \leqslant \beta < \frac{1}{2}.$$

Now, since $|1-r| \leqslant \rho$ and $|\sin(t/2)| \geqslant (t/\pi)$, then using Lemma 2.3 we have

$$|\eta_1| \leqslant \int_0^{a_n} \frac{|\varphi_x(t)|}{t} \left\{\left(n+\frac{1}{2}\right)t + (n+1)\left(ct^3 + \frac{rt}{1-r}\right)\right\} dt.$$

Also, since $t^3 \leqslant t$, then

$$|\eta_1| \leqslant C'n \int_0^{a_n} |\varphi_x(t)|\, dt$$

$$= C''n(a_n^{1+\alpha})$$

$$= 0(n^{-\alpha}).$$

We next consider

$$\eta_3 = \frac{1}{\pi}\int_{b_n}^{\pi} \frac{\varphi_x(t)}{\sin\frac{t}{2}} \left(\frac{1-r}{\rho}\right)^{n+1} \sin\left\{\left(n+\frac{1}{2}\right)t + (n+1)\theta\right\} dt.$$

By Lemma 2.2, $(1 - r/\rho)^n \leqslant e^{-Ant^2}$, $0 \leqslant t \leqslant \pi$, so that

$$
\begin{aligned}
|\eta_3| &\leqslant \frac{1}{\pi} \cdot \pi b_n^{-1} \int_{b_n}^{\pi} |\varphi_x(t)|\, e^{-A(n+1)t^2}\, dt \\
&\leqslant b_n^{-1} e^{-A(n+1)b_n^2} \int_0^{\pi} |\varphi_x(t)|\, dt \\
&\leqslant C n^{\beta} e^{-A' n^{1-2\beta}}, \qquad A' \text{ constant,} \\
&= 0(n^{-\alpha}) \qquad \text{for} \quad \beta < \tfrac{1}{2}.
\end{aligned}
$$

Finally, we study η_2 by writing

$$
\begin{aligned}
\eta_2 &= \frac{1}{\pi} \int_{a_n}^{b_n} \frac{\varphi_x(t)}{\sin \dfrac{t}{2}} \left(\frac{1-r}{\rho} \right)^{n+1} \sin \left\{ \left(n + \frac{1}{2} \right) t + (n+1)\, \theta \right\} dt \\
&= \frac{2}{\pi} \int_{a_n}^{b_n} \frac{\varphi_x(t)}{t} e^{-nrt^2/2(1-r)^2} \sin \left\{ \left(n + \frac{1}{2} \right) t + (n+1)\, \theta \right\} dt \\
&\quad + \frac{1}{\pi} \int_{a_n}^{b_n} \varphi_x(t) \left[\frac{1}{\sin \dfrac{t}{2}} \left(\frac{1-r}{\rho} \right)^{n+1} \right. \\
&\quad \left. - \frac{1}{\dfrac{t}{2}} e^{-nrt^2/2(1-r)^2} \right] \sin \left\{ \left(n + \frac{1}{2} \right) t + (n+1)\, \theta \right\} dt \\
&= \mu_1 + \mu_2, \qquad \text{say.}
\end{aligned}
$$

Now we have

$$
\begin{aligned}
|\mu_2| &\leqslant \int_{a_n}^{b_n} |\varphi_x(t)| \frac{2}{t} \left\{ \left(\frac{1-r}{\rho} \right)^{n+1} - e^{-nrt^2/2(1-r)^2} \right\} dt \\
&\quad + \int_{a_n}^{b_n} |\varphi_x(t)| \left| \frac{2}{t} - \frac{1}{\sin \dfrac{t}{2}} \right| \left(\frac{1-r}{\rho} \right)^{n+1} dt \\
&= \nu_1 + \nu_2,
\end{aligned}
$$

say, where by Lemma 2.2

$$
\begin{aligned}
|\nu_1| &\leqslant 2 \int_{a_n}^{b_n} \frac{|\varphi_x(t)|}{t} B(n+1)\, t^4\, dt, \\
&\leqslant C \cdot n n^{-3\beta} n^{-(1+\alpha)\beta} \\
&= 0(n^{-\alpha}).
\end{aligned}
$$

Also,

$$
\begin{aligned}
|v_2| &\leqslant C' \int_{a_n}^{b_n} t\, |\varphi_x(t)|\, dt \\
&\leqslant C'' n^{\beta} n^{-\beta(1+\alpha)} \\
&= 0(n^{-\alpha}) \qquad \text{by hypothesis.}
\end{aligned}
$$

Finally, we write

$$
\begin{aligned}
\mu_1 = {} & \frac{2}{\pi} \int_{a_n}^{b_n} \frac{\varphi_x(t)}{t} e^{-nrt^2/2(1-r)^2} \sin\left(n + \frac{1}{2} + \frac{n+1}{1-r} \cdot r\right) t\, dt \\
& + \frac{2}{\pi} \int_{a_n}^{b_n} \frac{\varphi_x(t)}{t} e^{-nrt^2/2(1-r)^2} \\
& \times \left| \sin \left\{ \left(n + \frac{1}{2}\right) t + (n+1)\, \theta \right\} \right. \\
& \left. - \sin\left(n + \frac{1}{2} + \frac{n+1}{1-r} \cdot r\right) t \right| dt \\
= {} & \chi_1 + \chi_2, \qquad \text{say,}
\end{aligned}
$$

where

$$
\begin{aligned}
|\chi_2| &\leqslant \frac{2}{\pi} \int_{a_n}^{b_n} \frac{|\varphi_x(t)|}{t} (n+1) \left| \theta - \frac{rt}{1-r} \right| dt \\
&\leqslant \frac{2C}{\pi} \int_{a_n}^{b_n} \frac{|\varphi_x(t)|}{t} (n+1)\, t^3\, dt, \qquad \text{by Lemma 2.3,} \\
&= 0(n^{-\alpha}), \qquad \text{by hypothesis.}
\end{aligned}
$$

and since

$$
a_n = \pi \left\{ n + \frac{1}{2} + \frac{n+1}{1-r} \cdot r \right\}^{-1},
$$

then

$$
\begin{aligned}
\chi_1 = {} & -\frac{2}{\pi} \int_0^{b_n - a_n} \frac{\varphi_x(t + a_n)}{t + a_n} e^{-nr(t+a_n)^2/2(1-r)^2} \\
& \times \sin\left(n + \frac{1}{2} + \frac{n+1}{1-r} \cdot r\right) t\, dt.
\end{aligned}
$$

Taking the average, we have that

$$\begin{aligned}\chi_1 &= \frac{1}{\pi}\int_{a_n}^{b_n} \frac{\varphi_x(t) - \varphi_x(t+a_n)}{t} e^{-nrt^2/2(1-r)^2} \sin(a^{-1}\pi t)\, dt \\ &+ \frac{1}{\pi}\int_{a_n}^{b_n} \frac{\varphi_x(t+a_n)}{t} |e^{-nrt^2/2(1-r)^2} \\ &- e^{-nr(t+a_n)^2/2(1-r)^2}| \sin(a_n^{-1}\pi t)\, dt \\ &= 0(n^{-\alpha})\end{aligned}$$

by the same method as used in Part I. Hence

$$\chi_1 = 0(n^{-\alpha})$$

and thus

$$\|\sigma_n^r - f(x)\| = 0(n^{-\alpha}). \quad \blacksquare$$

3. Remarks: (1) We believe that $0(n^{-\alpha})$ is the optimal order and that Taylor means are saturated with order $0(n^{-\alpha})$. Also, for $\alpha = 1$, the order of error is believed to be $0\{\log n/n^{\alpha}\}$.

(2) Large "O" of the theorem may again be replaced by small "o," if in the hypotheses of the theorem we replace large "O" by small "o" and Lip α by lip α (the set of functions f satisfying $|f(x+h) - f(x)| = o(|h|^{\alpha})$, uniformly in x).

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